Formulaire sur les comparaisons de fonctions

1 Règles de calculs

$$\left.\begin{array}{l}
\text{Si } \lim\limits_{x\longrightarrow a} \dfrac{f(x)}{g(x)} = 0, \text{ alors } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)).\\
\text{Si } \dfrac{f(x)}{g(x)} \text{ est borné, alors } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)).\\
\text{Si } \lim\limits_{x\longrightarrow a} \dfrac{f(x)}{g(x)} = 1, \text{ alors } f(x) \underset{x\to a}{\sim} (g(x)).
\end{array}\right\}\begin{array}{c}\text{Caractérisation}\\ \text{fondamentale}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)) \text{ et } g(x) = \underset{x\to a}{\mathrm{o}}(h(x)), \text{ alors } f(x) = \underset{x\to a}{\mathrm{o}}(h(x)).\\
\text{Si } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)) \text{ et } g(x) = \underset{x\to a}{\mathrm{O}}(h(x)), \text{ alors } f(x) = \underset{x\to a}{\mathrm{O}}(h(x)).\\
\text{Si } f(x) \underset{x\to a}{\sim} g(x) \text{ et } g(x) \underset{x\to a}{\sim} h(x), \text{ alors } f(x) \underset{x\to a}{\sim} h(x).
\end{array}\right\}\text{Transitivité}$$

$$\left.\begin{array}{l}
\text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)), \text{ alors } h(x)f(x) = \underset{x\to a}{\mathrm{o}}(h(x)g(x)).\\
\text{Si } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)), \text{ alors } h(x)f(x) = \underset{x\to a}{\mathrm{O}}(h(x)g(x)).\\
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } h(x)f(x) \underset{x\to a}{\sim} h(x)g(x).
\end{array}\right\}\begin{array}{c}\text{Compatibilité}\\ \text{avec le produit}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } \left.\begin{array}{l} f_1(x) = \underset{x\to a}{\mathrm{o}}(g_1(x))\\ f_2(x) = \underset{x\to a}{\mathrm{o}}(g_2(x))\end{array}\right\}, \text{ alors } f_1(x)f_2(x) = \underset{x\to a}{\mathrm{o}}(g_1(x)g_2(x)).\\
\text{Si } \left.\begin{array}{l} f_1(x) = \underset{x\to a}{\mathrm{O}}(g_1(x))\\ f_2(x) = \underset{x\to a}{\mathrm{O}}(g_2(x))\end{array}\right\}, \text{ alors } f_1(x)f_2(x) = \underset{x\to a}{\mathrm{O}}(g_1(x)g_2(x)).\\
\text{Si } \left.\begin{array}{l} f_1(x) \underset{x\to a}{\sim} g(x)\\ f_2(x) \underset{x\to a}{\sim} g(x)\end{array}\right\}, \text{ alors } f_1(x)f_2(x) \underset{x\to a}{\sim} g_1(x)g_2(x).
\end{array}\right\}\begin{array}{c}\text{Compatibilité}\\ \text{avec le produit}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)), \text{ alors } \dfrac{1}{g(x)} = \underset{x\to a}{\mathrm{o}}\left(\dfrac{1}{f(x)}\right).\\
\text{Si } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)), \text{ alors } \dfrac{1}{g(x)} = \underset{x\to a}{\mathrm{O}}\left(\dfrac{1}{f(x)}\right).\\
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } \dfrac{1}{f(x)} \underset{x\to a}{\sim} \dfrac{1}{g(x)}.
\end{array}\right\}\text{Règle d'inversion}$$

$$\left.\begin{array}{l}
\text{Si } n \in \mathbb{N}^* \text{ et } \alpha \in \mathbb{R}^{+*}, \text{ alors :}\\
\quad \text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)), \text{ alors } \left\{\begin{array}{l} f^n(x) = \underset{x\to a}{\mathrm{o}}(g^n(x))\\ f^\alpha(x) = \underset{x\to a}{\mathrm{o}}(g^\alpha(x))\end{array}\right.\\
\quad \text{Si } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)), \text{ alors } \left\{\begin{array}{l} f^n(x) = \underset{x\to a}{\mathrm{O}}(g^n(x))\\ f^\alpha(x) = \underset{x\to a}{\mathrm{O}}(g^\alpha(x))\end{array}\right.\\
\quad \text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } \left\{\begin{array}{l} f^n(x) \underset{x\to a}{\sim} g^n(x)\\ f^\alpha(x) \underset{x\to a}{\sim} g^\alpha(x)\end{array}\right.
\end{array}\right\}\begin{array}{c}\text{Compatibilité}\\ \text{avec les puissances.}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } \left.\begin{array}{l} f_1(x) = \underset{x\to a}{\mathrm{o}}(g(x))\\ f_2(x) = \underset{x\to a}{\mathrm{o}}(g(x))\end{array}\right\}, \text{ alors } \lambda f_1(x) + \mu f_2(x) = \underset{x\to a}{\mathrm{o}}(g(x)).\\
\text{Si } \left.\begin{array}{l} f_1(x) = \underset{x\to a}{\mathrm{O}}(g(x))\\ f_2(x) = \underset{x\to a}{\mathrm{O}}(g(x))\end{array}\right\}, \text{ alors } \lambda f_1(x) + \mu f_2(x) = \underset{x\to a}{\mathrm{O}}(g(x)).\\
\text{Si } \left.\begin{array}{l} f_1(x) \underset{x\to a}{\sim} g(x)\\ f_2(x) \underset{x\to a}{\sim} g(x)\\ \underline{\lambda + \mu \neq 0}\end{array}\right\}, \text{ alors } \lambda f_1(x) + \mu f_2(x) \underset{x\to a}{\sim} (\lambda + \mu)g(x).
\end{array}\right\}\begin{array}{c}\text{Combinaisons}\\ \text{linaires}\end{array}$$

$$\left.\begin{array}{l}
\underline{\text{Si } \lim\limits_{x\longrightarrow b} \varphi(x) = a}, \text{ alors :}\\
\quad \text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)), \text{ alors } f(\varphi(x)) = \underset{x\to b}{\mathrm{o}}(g(\varphi(x))).\\
\quad \text{Si } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)), \text{ alors } f(\varphi(x)) = \underset{x\to b}{\mathrm{O}}(g(\varphi(x))).\\
\quad \text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } f(\varphi(x)) \underset{x\to b}{\sim} g(\varphi(x)).
\end{array}\right\}\begin{array}{c}\text{Composition}\\ \text{à droite}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)), \text{ alors } f(x) = \underset{x\to a}{\mathrm{O}}(g(x))\\
\text{Si } f(x) = \underset{x\to a}{\mathrm{o}}(g(x)) \text{ et } g(x) = \underset{x\to a}{\mathrm{O}}(h(x)), \text{ alors } f(x) = \underset{x\to a}{\mathrm{o}}(h(x)).\\
\text{Si } f(x) = \underset{x\to a}{\mathrm{O}}(g(x)) \text{ et } g(x) = \underset{x\to a}{\mathrm{o}}(h(x)), \text{ alors } f(x) = \underset{x\to a}{\mathrm{o}}(h(x)).
\end{array}\right\}\begin{array}{c}\text{Liens entre}\\ \underset{x\longrightarrow a}{\mathrm{o}} \text{ et } \underset{x\longrightarrow a}{\mathrm{O}}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } f(x) = \underset{x\to a}{\mathrm{o}}(h(x)) \iff g(x) = \underset{x\to a}{\mathrm{o}}(h(x)).\\
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } h(x) = \underset{x\to a}{\mathrm{o}}(f(x)) \iff h(x) = \underset{x\to a}{\mathrm{o}}(g(x)).\\
f(x) \underset{x\to a}{\sim} g(x) \iff f(x) = g(x) + \underset{x\to a}{\mathrm{o}}(g(x))
\end{array}\right\}\begin{array}{c}\text{Liens entre}\\ \underset{x\longrightarrow a}{o} \text{ et } \underset{x\to a}{\sim}\end{array}$$

$$\left.\begin{array}{l}
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } f(x) = \underset{x\to a}{\mathrm{O}}(g(x))\\
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } f(x) = \underset{x\to a}{\mathrm{O}}(h(x)) \iff g(x) = \underset{x\to a}{\mathrm{O}}(h(x)).\\
\text{Si } f(x) \underset{x\to a}{\sim} g(x), \text{ alors } h(x) = \underset{x\to a}{\mathrm{O}}(f(x)) \iff h(x) = \underset{x\to a}{\mathrm{O}}(g(x)).
\end{array}\right\}\begin{array}{c}\text{Liens entre}\\ \underset{x\to a}{\sim} \text{ et } \underset{x\longrightarrow a}{\mathrm{O}}\end{array}$$

2 Fonctions négligeables

Si $(n,m) \in \mathbb{Z}^2$ et $n > m$, alors

1. $x^n \underset{x\to 0}{=} \mathrm{o}(x^m)$
2. $x^m \underset{x\to +\infty}{=} \mathrm{o}(x^n)$
3. $x^m \underset{x\to -\infty}{=} \mathrm{o}(x^n)$

Si $(\alpha,\beta) \in \mathbb{R}^2$ et $\alpha > \beta$, alors

1. $x^\alpha \underset{x\to 0}{=} \mathrm{o}\left(x^\beta\right)$
2. $x^\beta \underset{x\to +\infty}{=} \mathrm{o}(x^\alpha)$

Si $(\alpha,\beta) \in \mathbb{R}^3$ tel que $\alpha > 0$, alors

1. $x^\beta \underset{x\to +\infty}{=} \mathrm{o}(\mathrm{e}^x)$,
   $x^\beta \underset{x\to +\infty}{=} \mathrm{o}(\mathrm{e}^{\alpha x})$
2. $\ln(x) \underset{x\to \infty}{=} \mathrm{o}(x^\alpha)$,
   $(\ln(x))^\beta \underset{x\to +\infty}{=} \mathrm{o}(x^\alpha)$
3. $x^\alpha \underset{x\to 0}{=} \mathrm{o}(\ln(x))$,
   $\ln(x) \underset{x\to 0}{=} \mathrm{o}\left(x^{-\alpha}\right)$,
   $(\ln(x))^\beta \underset{x\to 0}{=} \mathrm{o}\left(x^{-\alpha}\right)$

3 Fonctions équivalentes

Lien avec les limites :
Si $f(x) \underset{x\to a}{\sim} g(x)$ et $\lim\limits_{x\longrightarrow a} g(x) = \ell$, alors :

$$\lim_{x\longrightarrow a} f(x) = \ell$$

Si $\underline{\ell \neq 0}$, alors :

$$\lim_{x\longrightarrow a} f(x) = \ell \iff f(x) \underset{x\to a}{\sim} \ell$$

Fonctions dérivables :
Si f est dérivable en a et $\underline{f'(a) \neq 0}$, alors :

$$f(x) - f(a) \underset{x\to a}{\sim} f'(a)(x-a)$$

Fonctions usuelles :

$$\begin{array}{rcl} \sin(x) \underset{x\to 0}{\sim} x & , & \arcsin(x) \underset{x\to 0}{\sim} x \\ \tan(x) \underset{x\to 0}{\sim} x & , & \arctan(x) \underset{x\to 0}{\sim} x \\ \mathrm{sh}(x) \underset{x\to 0}{\sim} x & , & \mathrm{Argsh}(x) \underset{x\to 0}{\sim} x \\ \mathrm{th}(x) \underset{x\to 0}{\sim} x & , & \mathrm{Argth}(x) \underset{x\to 0}{\sim} x \\ \ln(1+x) \underset{x\to 0}{\sim} x & , & e^x - 1 \underset{x\to 0}{\sim} x \\ 1-\cos(x) \underset{x\to 0}{\sim} \dfrac{x^2}{2} & , & \mathrm{ch}(x) - 1 \underset{x\to 0}{\sim} \dfrac{x^2}{2} \\ (1+x)^\alpha - 1 \underset{x\to 0}{\sim} \alpha x, & & \end{array}$$

Polynômes :
Si $P(x) = a_p x^p + a_{p+1}x^{p+1} + \cdots + a_n x^n$ avec $a_p \neq 0$, $a_n \neq 0$ et $p \leq n$, alors

1. $P(x) \underset{x\to 0}{\sim} a_p x^p$ (terme de plus bas degré)
2. $P(x) \underset{x\to +\infty}{\sim} a_n x^n$ (terme de plus haut degré)
3. $P(x) \underset{x\to -\infty}{\sim} a_n x^n$ (terme de plus haut degré)

Fractions rationnelles :
Si $F(x) = \dfrac{a_p x^p + a_{p+1}x^{p+1} + \cdots + a_n x^n}{b_q x^q + b_{q+1}x^{q+1} + \cdots + b_m x^m}$ avec $a_p \neq 0$,$a_n \neq 0$, $b_q \neq 0$, $b_m \neq 0$, $p \leq n$ et $q \leq m$, alors

1. $F(x) \underset{x\to 0}{\sim} \dfrac{a_p x^p}{b_q x^q}$
2. $F(x) \underset{x\to +\infty}{\sim} \dfrac{a_n x^n}{b_m x^m}$
3. $F(x) \underset{x\to -\infty}{\sim} \dfrac{a_n x^n}{b_m x^m}$